Contents

This analysis looks at deriving the equation for the electric from a charge moving with a velocity in an arbitrary direction. We are ultimately looking to derive:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{\bar{e_r}}{r^2} + \frac{r}{c}\frac{d}{dt}\left(\frac{\bar{e_r}}{r^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\bar{e_r}\right]$$

1 Background

The problem is shown in Figure 1.

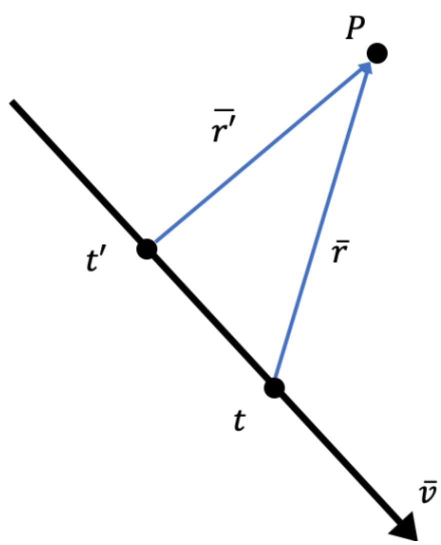

Figure 1: The electric field at the time, t, depends on the particle at the time, t', where $t = t' + \frac{r'}{c}$

We see that the particle is moving with velocity, $\bar{v}$. The charge doesn't necessarily have to be moving in a straight line but we will assume that the velocity is constant. We are interested in the electric field at the point, P, at time, t. This means we need to know where the particle is at the retarded time, t'. This is because the point P is influenced by the charge at time t' because $t = t' + \frac{r'}{c}$.

We follow Feynman's derivation of the scalar and vector potential, and start with:

$$\phi(P,t) = \frac{q}{4\pi\epsilon_o\left[r - \left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]_{ret}}$$

$$\bar{A}(P,t) = \frac{q\bar{v}}{4\pi\epsilon_o c^2\left[r - \frac{(\bar{v}\cdot\bar{r})}{c}\right]_{ret}}$$

where q is the charge of the moving particle, ϵ_o is the permittivity of free space, r' is the distance from the particle to point P at the retarded time t', v_r is the velocity of the particle (which we will make assumptions about), and c is the speed of sound. The terms in the bracket are evaluated at the retarded time, t'.

We also know that the electric field is:

$$\bar{E} = -\nabla\phi - \frac{\partial\bar{A}}{\partial t}$$

Looking at the electric field equation, we see that the space derivative and time derivative are at point P at time t but this will also depend on r' and therefore, will also depend on t'

We'll first find the space and time derivative of r', and then combine this with the electric field equation above.

2 Space derivative of $\bar{r}$

We'll find the space derivative of $\bar{r}'$ here. We'll first consider the derivative with respect to x. We start with Figure 2 below.

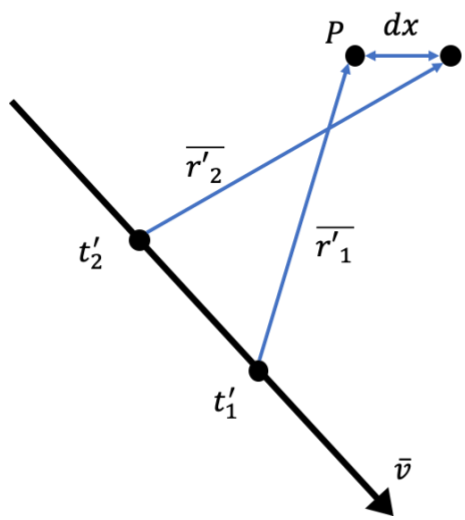

Figure 2: Schematic for finding space derivative of $\bar{r}$

We know that:

$$t = t'_1 + \frac{r'_1}{c} = t'_2 + \frac{r'_2}{c}$$

We also know that:

$$\overline{r'_2} - \overline{r'_1} = dx + \bar{v}(t'_1 - t'_2)$$

We know that $t'_1 > t'_2$ since r'_2 must be longer than r'_1 (imagine drawing a circle of radius r'_1 from the point that particle is at t'_1)

From the first equation, we have:

$$t'_1 - t'_2 = \frac{1}{c}(r'_2 - r'_1)$$

Plugging this into the second equation gives:

$$\overline{r'}_2 - \overline{r'}_1 = dx + \frac{\bar{v}}{c}(r'_2 - r'_1)$$

We can generalize from dx to $\overline{ds}$ with no loss (all previous assumptions hold), so we get:

$$\overline{r'}_2 - \overline{r'}_1 = \overline{ds} + \frac{\bar{v}}{c}(r'_2 - r'_1)$$

We then look at small changes and say $\overline{r_2} = \overline{r_1} + \overline{dr}$ (we'll stop using the primes knowing that we are talking about the retarded time. So now we have:

$$\overline{r_2} - \overline{r_1} = \overline{dr} = \overline{ds} + \frac{\bar{v}}{c}(r'_2 - r'_1) = \overline{ds} + \frac{\bar{v}}{c}\left(\sqrt{\left(\overline{r_1} + \overline{dr}\right)^2} - \sqrt{(\overline{r_1})^2}\right)$$

We can focus on the second term (term in parentheses) on the RHS of the last equality:

$$\sqrt{\left(\overline{r_1} + \overline{dr}\right)^2} - \sqrt{(\overline{r_1})^2} = \sqrt{(\overline{r_1})^2 + 2\overline{r_1}\cdot\overline{dr} + \left(\overline{dr}\right)^2} - \sqrt{(\overline{r_1})^2} \cong \sqrt{(\overline{r_1})^2 + 2\overline{r_1}\cdot\overline{dr}} - \sqrt{(\overline{r_1})^2}$$

In the last equality, we assume $\left(\overline{dr}\right)^2$ is small. We use the binomial approximation to further simplify:

$$\sqrt{(\overline{r_1})^2 + 2\overline{r_1}\cdot\overline{dr}} - \sqrt{(\overline{r_1})^2} = r_1\sqrt{1 + \frac{2\overline{r_1}\cdot\overline{dr}}{r_1}} - r_1 \cong r_1\left(1 + \frac{1}{2}\frac{2\overline{r_1}\cdot\overline{dr}}{r_1}\right) - r_1 = \frac{\overline{r_1}\cdot\overline{dr}}{r_1}$$

We then (and dropping the subscript):

$$\overline{dr} = \overline{ds} + \frac{\bar{v}}{c}\left(\frac{\bar{r}\cdot\overline{dr}}{r}\right)$$

To understand this, it's easier to look at the components:

$$\left(dr_x, dr_y, dr_z\right) = (dx, dy, dz) + \frac{\bar{v}}{c}\left(\frac{r_x dr_x + r_y dr_y + r_z dr_z}{r}\right)$$

So then we have:

$$dr_x = dx + \frac{v_x}{c}\left(\frac{r_x dr_x + r_y dr_y + r_z dr_z}{r}\right)$$

$$dr_y = dy + \frac{v_y}{c}\left(\frac{r_x dr_x + r_y dr_y + r_z dr_z}{r}\right)$$

$$dr_z = dz + \frac{v_z}{c}\left(\frac{r_x dr_x + r_y dr_y + r_z dr_z}{r}\right)$$

We will look at finding $\frac{dr_x}{dx}$ first:

$$\frac{dr_x}{dx} = 1 + \frac{v_x}{c}\left(\frac{r_x\frac{dr_x}{dx} + r_y\frac{dr_y}{dx} + r_z\frac{dr_z}{dx}}{r}\right)$$

Now we need to find $\frac{dr_y}{dx}$ and $\frac{dr_z}{dx}$. To find $\frac{dr_y}{dx}$ and $\frac{dr_z}{dx}$, we start with:

$$\frac{dr_y}{dx} = \frac{dy}{dx} + \frac{v_y}{c}\left(\frac{r_x\frac{dr_x}{dx} + r_y\frac{dr_y}{dx} + r_z\frac{dr_z}{dx}}{r}\right)$$

$$\frac{dr_z}{dx} = \frac{dz}{dx} + \frac{v_z}{c}\left(\frac{r_x \frac{dr_x}{dx} + r_y \frac{dr_y}{dx} + r_z \frac{dr_z}{dx}}{r}\right)$$

$\frac{dy}{dx}$ and $\frac{dz}{dx}$ are zero so these will drop out. Grouping like terms ad solving for $\frac{dr_y}{dx}$ and $\frac{dr_z}{dx}$ gives:

$$\frac{dr_y}{dx}\left(1 - \frac{v_y r_y}{cr}\right) = \frac{v_y}{c}\left(\frac{r_x \frac{dr_x}{dx} + r_z \frac{dr_z}{dx}}{r}\right)$$

$$\frac{dr_z}{dx}\left(1 - \frac{v_z r_z}{cr}\right) = \frac{v_z}{c}\left(\frac{r_x \frac{dr_x}{dx} + r_y \frac{dr_y}{dx}}{r}\right)$$

Multiplying the last equation by $\left(1 - \frac{v_y r_y}{cr}\right)$ and substituting gives:

$$\frac{dr_z}{dx}\left(1 - \frac{v_z r_z}{cr}\right)\left(1 - \frac{v_y r_y}{cr}\right) = \frac{v_z}{c}\left(\frac{r_x \frac{dr_x}{dx}\left(1 - \frac{v_y r_y}{cr}\right) + r_y \frac{dr_y}{dx}\left(1 - \frac{v_y r_y}{cr}\right)}{r}\right)$$

$$= \frac{v_z}{c}\left(\frac{r_x \frac{dr_x}{dx}\left(1 - \frac{v_y r_y}{cr}\right) + r_y \frac{v_y}{c}\left(\frac{r_x \frac{dr_x}{dx} + r_z \frac{dr_z}{dx}}{r}\right)}{r}\right)$$

We can simplify this to get:

$$\frac{dr_z}{dx}\left(1 - \frac{v_z r_z}{cr}\right)\left(1 - \frac{v_y r_y}{cr}\right) = \frac{v_z}{c}\left(\frac{r_x \frac{dr_x}{dx}\left(1 - \frac{v_y r_y}{cr}\right) + r_y \frac{v_y}{c}\left(\frac{r_x \frac{dr_x}{dx} + r_z \frac{dr_z}{dx}}{r}\right)}{r}\right)$$

$$\frac{dr_z}{dx}\left(1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr} + \frac{v_y r_y v_z r_z}{c^2 r^2}\right) = \frac{v_z}{cr}\left(r_x \frac{dr_x}{dx} + \frac{r_z v_y r_y}{cr}\frac{dr_z}{dx}\right)$$

$$\frac{dr_z}{dx}\left(1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}\right) = \frac{v_z}{cr} r_x \frac{dr_x}{dx}$$

So then we obtain:

$$\frac{dr_z}{dx} = \frac{\frac{v_z}{cr} r_x}{1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}}\frac{dr_x}{dx}$$

We do the same thing, but for $\frac{dr_y}{dx}$, we multiply by $\left(1 - \frac{v_z r_z}{cr}\right)$ and follow the same procedure:

$$\frac{dr_y}{dx}\left(1 - \frac{v_y r_y}{cr}\right)\left(1 - \frac{v_z r_z}{cr}\right) = \frac{v_y}{c}\left(\frac{r_x\left(1 - \frac{v_z r_z}{cr}\right)\frac{dr_x}{dx} + r_z\left(1 - \frac{v_z r_z}{cr}\right)\frac{dr_z}{dx}}{r}\right)$$

We can substitute in for $\left(1 - \frac{v_z r_z}{cr}\right)\frac{dr_z}{dx}$. We obtain:

$$\frac{dr_y}{dx}\left(1-\frac{v_y r_y}{cr}\right)\left(1-\frac{v_z r_z}{cr}\right)=\frac{v_y}{c}\left(\frac{r_x\left(1-\frac{v_z r_z}{cr}\right)\frac{dr_x}{dx}+r_z\frac{v_z}{c}\left(\frac{r_x\frac{dr_x}{dx}+r_y\frac{dr_y}{dx}}{r}\right)}{r}\right)$$

Expanding this out we get and simplifying:

$$\frac{dr_y}{dx}\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}+\frac{v_y r_y v_z r_z}{c^2 r^2}\right)=\frac{v_y}{cr}\left(r_x\frac{dr_x}{dx}+\frac{r_y v_z r_z}{cr}\frac{dr_y}{dx}\right)$$

$$\frac{dr_y}{dx}=\frac{\frac{v_y}{cr}r_x}{1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}}\frac{dr_x}{dx}$$

So we know have:

$$\frac{dr_x}{dx}=1+\frac{v_x}{c}\left(\frac{r_x\frac{dr_x}{dx}+r_y\frac{dr_y}{dx}+r_z\frac{dr_z}{dx}}{r}\right)$$

$$\frac{dr_y}{dx}=\frac{\frac{v_y}{cr}r_x}{1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}}\frac{dr_x}{dx}$$

$$\frac{dr_z}{dx}=\frac{\frac{v_z}{cr}r_x}{1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}}\frac{dr_x}{dx}$$

Substituting in gives:

$$\frac{dr_x}{dx}=1+\frac{v_x}{cr}\left(r_x\frac{dr_x}{dx}+\frac{\frac{v_y r_y}{cr}r_x}{1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}}\frac{dr_x}{dx}+\frac{\frac{v_z r_z}{cr}r_x}{1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}}\frac{dr_x}{dx}\right)$$

Multiplying by $\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)$ gives:

$$\frac{dr_x}{dx}\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)=\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)+\frac{v_x}{cr}\left[\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)r_x\frac{dr_x}{dx}+\frac{v_y r_y}{cr}r_x\frac{dr_x}{dx}+\frac{v_z r_z}{cr}r_x\frac{dr_x}{dx}\right]$$

$$\frac{dr_x}{dx}\left(1-\frac{v_x r_x}{cr}-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)=\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)$$

$$\frac{dr_x}{dx}\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)=1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}$$

Finally, we get:

$$\frac{dr_x}{dx}=\frac{1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}=\frac{1-\frac{v_x r_x}{cr}-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}+\frac{v_x r_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}=1+\frac{\frac{v_x r_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

We want to find $\frac{dr_y}{dx}$ and then hopefully, we can generalize from $\frac{dr_x}{dx}$ and $\frac{dr_y}{dx}$. We have both $\frac{dr_y}{dx}$ and $\frac{dr_x}{dx}$ already:

$$\frac{dr_y}{dx} = \frac{\frac{v_y}{cr} r_x}{1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}} \frac{dr_x}{dx}$$

$$\frac{dr_x}{dx} = 1 + \frac{\frac{v_x r_x}{cr}}{1 - \frac{\bar{v} \cdot \bar{r}}{cr}}$$

Substituting in for $\frac{dr_x}{dx}$ gives:

$$\frac{dr_y}{dx} = \frac{\frac{v_y}{cr} r_x}{1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}} \left(1 + \frac{\frac{v_x r_x}{cr}}{1 - \frac{\bar{v} \cdot \bar{r}}{cr}}\right)$$

$$= \frac{\frac{v_y}{cr} r_x}{1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}} + \frac{\frac{v_y v_x r_x^2}{c^2 r^2}}{\left(1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}\right)\left(1 - \frac{\bar{v} \cdot \bar{r}}{cr}\right)}$$

We can multiply the first term on the RHS by $\left(1 - \frac{\bar{v} \cdot \bar{r}}{cr}\right) / \left(1 - \frac{\bar{v} \cdot \bar{r}}{cr}\right)$ and see if we can simplify.

$$\frac{dr_y}{dx} = \frac{\left(1 - \frac{\bar{v} \cdot \bar{r}}{cr}\right)\frac{v_y}{cr} r_x}{\left(1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}\right)\left(1 - \frac{\bar{v} \cdot \bar{r}}{cr}\right)} + \frac{\frac{v_y v_x r_x^2}{c^2 r^2}}{\left(1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}\right)\left(1 - \frac{\bar{v} \cdot \bar{r}}{cr}\right)}$$

We'll just look the numerator for now:

$$\left(1 - \frac{\bar{v} \cdot \bar{r}}{cr}\right)\frac{v_y}{cr} r_x + \frac{v_y v_x r_x^2}{c^2 r^2} = \frac{v_y}{cr} r_x - \frac{v_x v_y r_x^2}{c^2 r^2} - \frac{v_y^2 r_y r_x}{c^2 c^2} - \frac{r_x v_y v_z r_z}{c^2 r^2} + \frac{v_y v_x r_x^2}{c^2 r^2}$$

$$= \frac{v_y r_x}{cr}\left(1 - \frac{v_x r_x}{cr} - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr} + \frac{v_x r_x}{cr}\right) = \frac{v_y r_x}{cr}\left(1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}\right)$$

We then have:

$$\frac{dr_y}{dx} = \frac{\frac{v_y r_x}{cr}\left(1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}\right)}{\left(1 - \frac{v_y r_y}{cr} - \frac{v_z r_z}{cr}\right)\left(1 - \frac{\bar{v} \cdot \bar{r}}{cr}\right)} = \frac{\frac{v_y r_x}{cr}}{1 - \frac{\bar{v} \cdot \bar{r}}{cr}}$$

So then have:

$$\frac{dr_x}{dx} = 1 + \frac{\frac{v_x r_x}{cr}}{1 - \frac{\bar{v} \cdot \bar{r}}{cr}}$$

$$\frac{dr_y}{dx} = \frac{\frac{v_y r_x}{cr}}{1 - \frac{\bar{v} \cdot \bar{r}}{cr}}$$

We can generalize this to be:

$$\frac{\partial r_i}{\partial x_j} = \delta_{ij} + \frac{\frac{v_i r_j}{cr}}{1 - \frac{\bar{v} \cdot \bar{r}}{cr}}$$

3 Time derivative of $\bar{r}$

We'll find the time derivative of $\bar{r}'$ here. We start with the following equations:

$$t_1 = t_1' + \frac{r_1'}{c}$$

$$t_2 = t_2' + \frac{r_2'}{c}$$

$$\overline{r'_2} - \overline{r'_1} = -\bar{v}(t_2' - t_1')$$

We use Figure 3 for our understanding of the time derivative. We know that $t_2' > t_1'$ since we are looking at the time derivative, and at $t + dt$, the particle will have travelled further. We want to replace t' in the last equation, and use the first two equations to find a relationship between t' and t:

$$t_2' - t_1' = (t_2 - t_1) - \frac{1}{c}(r_2' - r_1')$$

Plugging this in gives:

$$\overline{r'_2} - \overline{r'_1} = -\bar{v}\left[(t_2 - t_1) - \frac{1}{c}(r_2' - r_1')\right]$$

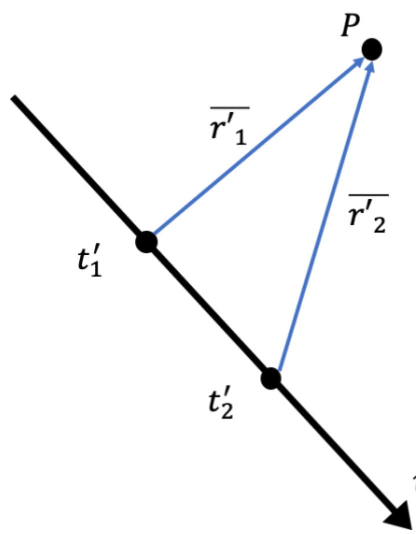

Figure 3: Schematic for finding time derivative of $\bar{r}$

We now have the equation we want to find the time derivative:

$$\overline{r'_2} - \overline{r'_1} = -\bar{v}\left[(t_2 - t_1) - \frac{1}{c}(r_2' - r_1')\right]$$

When we look at small changes, we have:

$$\overline{r'_2} - \overline{r'_1} = \overline{r'_1} + \overline{dr'} - \overline{r'_1} = \overline{dr'}$$

$$t_2 - t_1 = t_1 + dt - t_1 = dt$$

Using our approximation and simplification for $r_2' - r_1'$

$$r_2' - r_1' = \frac{\overline{r'_1} \cdot \overline{dr'}}{r'_1}$$

We then have (dropping subscripts and primes):

$$\overline{dr} = -\bar{v}\left(dt - \frac{\bar{r} \cdot \overline{dr}}{cr}\right)$$

Looking at the components, we have:

$$(dr_x, dr_y, dr_z) = -(v_x, v_y, v_z)\left(dt - \frac{r_x dr_x + r_y dr_y + r_z dr_z}{cr}\right)$$

Looking at the each component, we have:

$$dr_x = -v_x\left(dt - \frac{r_x dr_x + r_y dr_y + r_z dr_z}{cr}\right)$$

$$dr_y = -v_y\left(dt - \frac{r_x dr_x + r_y dr_y + r_z dr_z}{cr}\right)$$

$$dr_z = -v_z\left(dt - \frac{r_x dr_x + r_y dr_y + r_z dr_z}{cr}\right)$$

Looking the derivative with respect to time gives:

$$\frac{dr_x}{dt} = -v_x\left(1 - \frac{r_x \frac{dr_x}{dt} + r_y \frac{dr_y}{dt} + r_z \frac{dr_z}{dt}}{cr}\right)$$

$$\frac{dr_y}{dt} = -v_y\left(1 - \frac{r_x \frac{dr_x}{dt} + r_y \frac{dr_y}{dt} + r_z \frac{dr_z}{dt}}{cr}\right)$$

$$\frac{dr_z}{dt} = -v_z\left(1 - \frac{r_x \frac{dr_x}{dt} + r_y \frac{dr_y}{dt} + r_z \frac{dr_z}{dt}}{cr}\right)$$

We follow a similar procedure and start simplifying $\frac{dr_y}{dt}$ and $\frac{dr_z}{dt}$:

$$\frac{dr_y}{dt}\left(1 - \frac{v_y r_y}{cr}\right) = -v_y\left(1 - \frac{r_x \frac{dr_x}{dt} + r_z \frac{dr_z}{dt}}{cr}\right)$$

$$\frac{dr_z}{dt}\left(1 - \frac{v_z r_z}{cr}\right) = -v_z\left(1 - \frac{r_x \frac{dr_x}{dt} + r_y \frac{dr_y}{dt}}{cr}\right)$$

We can multiply the top equation by $\left(1 + \frac{v_z r_z}{cr}\right)$ and substitute:

$$\frac{dr_y}{dt}\left(1 - \frac{v_y r_y}{cr}\right)\left(1 - \frac{v_z r_z}{cr}\right) = -v_y\left(1 + \frac{v_z r_z}{cr} - \frac{r_x \frac{dr_x}{dt}\left(1 - \frac{v_z r_z}{cr}\right) + r_z \frac{dr_z}{dt}\left(1 - \frac{v_z r_z}{cr}\right)}{cr}\right)$$

$$= -v_y\left(1 + \frac{v_z r_z}{cr} - \frac{r_x \frac{dr_x}{dt}\left(1 - \frac{v_z r_z}{cr}\right) - v_z r_z\left(1 - \frac{r_x \frac{dr_x}{dt} + r_y \frac{dr_y}{dt}}{cr}\right)}{cr}\right)$$

$$= -v_y\left(1 + \frac{v_z r_z}{cr} - \frac{r_x \frac{dr_x}{dt} - v_z r_z + \frac{r_y v_z r_z}{cr}\frac{dr_y}{dt}}{cr}\right)$$

$$\frac{dr_y}{dt}\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}+\frac{v_y r_y v_z r_z}{c^2 r^2}-\frac{v_y r_y v_z r_z}{c^2 r^2}\right)=-v_y\left(1-\frac{r_x\frac{dr_x}{dt}}{cr}\right)$$

We follow a similar procedure for $\frac{dr_z}{dt}$:

$$\frac{dr_y}{dt}\left(1-\frac{v_y r_y}{cr}\right)=-v_y\left(1-\frac{r_x\frac{dr_x}{dt}+r_z\frac{dr_z}{dt}}{cr}\right)$$

$$\frac{dr_z}{dt}\left(1-\frac{v_z r_z}{cr}\right)=-v_z\left(1-\frac{r_x\frac{dr_x}{dt}+r_y\frac{dr_y}{dt}}{cr}\right)$$

We can multiply the bottom equation by $\left(1-\frac{v_y r_y}{cr}\right)$ and substitute:

$$\frac{dr_z}{dt}\left(1-\frac{v_z r_z}{cr}\right)\left(1-\frac{v_y r_y}{cr}\right)=-v_z\left(1-\frac{v_y r_y}{cr}-\frac{r_x\frac{dr_x}{dt}\left(1-\frac{v_y r_y}{cr}\right)+r_y\frac{dr_y}{dt}\left(1-\frac{v_y r_y}{cr}\right)}{cr}\right)$$

$$=-v_z\left(1-\frac{v_y r_y}{cr}-\frac{r_x\frac{dr_x}{dt}\left(1-\frac{v_y r_y}{cr}\right)-v_y r_y\left(1-\frac{r_x\frac{dr_x}{dt}+r_z\frac{dr_z}{dt}}{cr}\right)}{cr}\right)$$

$$=-v_z\left(1-\frac{v_y r_y}{cr}-\frac{r_x\frac{dr_x}{dt}-v_y r_y+\frac{v_y r_y r_z}{cr}\frac{dr_z}{dt}}{cr}\right)$$

$$\frac{dr_z}{dt}\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}+\frac{v_y r_y v_z r_z}{c^2 r^2}-\frac{v_y r_y v_z r_z}{c^2 r^2}\right)=-v_z\left(1-\frac{r_x\frac{dr_x}{dt}}{cr}\right)$$

We then have:

$$\frac{dr_x}{dt}=-v_x\left(1-\frac{r_x\frac{dr_x}{dt}+r_y\frac{dr_y}{dt}+r_z\frac{dr_z}{dt}}{cr}\right)$$

$$\frac{dr_y}{dt}\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)=-v_y\left(1-\frac{r_x\frac{dr_x}{dt}}{cr}\right)$$

$$\frac{dr_z}{dt}\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)=-v_z\left(1-\frac{r_x\frac{dr_x}{dt}}{cr}\right)$$

Substituting in gives:

$$\frac{dr_x}{dt}\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)$$

$$= -v_x\left[\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)-\frac{r_x\frac{dr_x}{dt}\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)-v_y r_y\left(1-\frac{r_x\frac{dr_x}{dt}}{cr}\right)-v_z r_z\left(1-\frac{r_x\frac{dr_x}{dt}}{cr}\right)}{cr}\right]$$

$$= -v_x\left(\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)-\frac{r_x\frac{dr_x}{dt}-v_y r_y-v_z r_z}{cr}\right)$$

$$= -v_x\left(1-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}-\frac{r_x}{cr}\frac{dr_x}{dt}+\frac{v_y r_y}{cr}+\frac{v_z r_z}{cr}\right)$$

We then get:

$$\frac{dr_x}{dt}\left(1-\frac{v_x r_x}{cr}-\frac{v_y r_y}{cr}-\frac{v_z r_z}{cr}\right)=-v_x$$

$$\frac{dr_x}{dt}=-\frac{v_x}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

We can generalize to:

$$\frac{dr_i}{dt}=-\frac{v_i}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

4 Electric field derivation

We know have:

$$\phi(P,t)=\frac{q}{4\pi\epsilon_o\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]_{ret}}$$

$$\bar{A}(P,t)=\frac{q\bar{v}}{4\pi\epsilon_o c^2\left[r-\frac{(\bar{v}\cdot\bar{r})}{c}\right]_{ret}}$$

$$\bar{E}=-\nabla\phi-\frac{\partial\bar{A}}{\partial t}$$

And just derived:

$$\frac{\partial r_i}{\partial x_j}=\delta_{ij}+\frac{\frac{v_i r_j}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

$$\frac{dr_i}{dt}=-\frac{v_i}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

4.1 Scalar potential

We'll first look at $\nabla\phi$. We drop primes and know when we are at looking at r, we are talking about r at the retarded time.

We now have:

$$\nabla\phi(P,t) = \nabla\frac{q}{4\pi\epsilon_o\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]_{ret}} = \frac{q}{4\pi\epsilon_o}\nabla\frac{1}{\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]_{ret}}$$

$$= \frac{q}{4\pi\epsilon_o}\left[-\frac{1}{\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]^2_{ret}}\left[\nabla r-\frac{1}{c}\nabla(\bar{v}\cdot\bar{r})\right]\right]$$

We have that $r=\sqrt{r_x^2+r_y^2+r_z^2}$ so that, for example, $\frac{\partial r}{\partial x}=\frac{r_x}{r}\left(1+\frac{\frac{v_xr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)+\frac{r_y}{r}\frac{\frac{v_yr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+\frac{r_z}{r}\frac{\frac{v_zr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$. We have then:

$$\nabla r=\left[\frac{r_x}{r}\left(1+\frac{\frac{v_xr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)+\frac{r_y}{r}\frac{\frac{v_yr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+\frac{r_z}{r}\frac{\frac{v_zr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}},\frac{r_x}{r}\frac{\frac{v_xr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+\frac{r_y}{r}\left(1+\frac{\frac{v_yr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)\right.$$
$$\left.+\frac{r_z}{r}\frac{\frac{v_zr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}},\frac{r_x}{r}\frac{\frac{v_xr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+\frac{r_y}{r}\frac{\frac{v_yr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+\frac{r_z}{r}\left(1+\frac{\frac{v_zr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)\right]=\frac{\bar{r}}{r}+\frac{\bar{r}}{r}\frac{\frac{\bar{v}\cdot\bar{r}}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

We have, from vector calculus, that:

$$\nabla(\bar{v}\cdot\bar{r})=\bar{v}\times(\nabla\times\bar{r})+\bar{r}\times(\nabla\times\bar{v})+(\bar{v}\cdot\nabla)\bar{r}+(\bar{r}\cdot\nabla)\bar{v}$$

This simplifies to (since we assumed $\bar{v}$ is constant):

$$\nabla(\bar{v}\cdot\bar{r})=\bar{v}\times(\nabla\times\bar{r})+(\bar{v}\cdot\nabla)\bar{r}$$

$$=\bar{v}\times\left[\left(\frac{\partial r_z}{\partial y}-\frac{\partial r_y}{\partial z}\right),-\left(\frac{\partial r_z}{\partial x}-\frac{\partial r_x}{\partial z}\right),\left(\frac{\partial r_y}{\partial x}-\frac{\partial r_x}{\partial y}\right)\right]$$
$$+\left[\left(v_x\frac{\partial r_x}{\partial x}+v_y\frac{\partial r_x}{\partial y}+v_z\frac{\partial r_x}{\partial z}\right),\left(v_x\frac{\partial r_y}{\partial x}+v_y\frac{\partial r_y}{\partial y}+v_z\frac{\partial r_y}{\partial z}\right),\left(v_x\frac{\partial r_z}{\partial x}+v_y\frac{\partial r_z}{\partial y}+v_z\frac{\partial r_z}{\partial z}\right)\right]$$

$$=\bar{v}\times\left[\left(\frac{\frac{v_zr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\frac{v_yr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right),-\left(\frac{\frac{v_zr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\frac{v_xr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right),\left(\frac{\frac{v_yr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\frac{v_xr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)\right]$$
$$+\left[\left(v_x\left(1+\frac{\frac{v_xr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)+v_y\frac{\frac{v_xr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+v_z\frac{\frac{v_xr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right),\left(v_x\frac{\frac{v_yr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+v_y\left(1+\frac{\frac{v_yr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)\right.\right.$$
$$\left.\left.+v_z\frac{\frac{v_yr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right),\left(v_x\frac{\frac{v_zr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+v_y\frac{\frac{v_zr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+v_z\left(1+\frac{\frac{v_zr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)\right)\right]$$

$$=\left[v_y\left(\frac{\frac{v_yr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\frac{v_xr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)+v_z\left(\frac{\frac{v_zr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\frac{v_xr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right),-v_x\left(\frac{\frac{v_yr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\frac{v_xr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)\right.$$
$$\left.+v_z\left(\frac{\frac{v_zr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\frac{v_yr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right),-v_x\left(\frac{\frac{v_zr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\frac{v_xr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)-v_y\left(\frac{\frac{v_zr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\frac{v_yr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)\right]$$
$$+\left[\left(v_x\left(1+\frac{\frac{v_xr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)+v_y\frac{\frac{v_xr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+v_z\frac{\frac{v_xr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right),\left(v_x\frac{\frac{v_yr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+v_y\left(1+\frac{\frac{v_yr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)\right.\right.$$
$$\left.\left.+v_z\frac{\frac{v_yr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right),\left(v_x\frac{\frac{v_zr_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+v_y\frac{\frac{v_zr_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+v_z\left(1+\frac{\frac{v_zr_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)\right)\right]$$

$$= \left(\frac{\frac{v_y v_y r_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + \frac{\frac{v_z v_z r_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + v_x + \frac{\frac{v_x v_x r_x}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}, \frac{\frac{v_x v_x r_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + \frac{\frac{v_z v_z r_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + v_y + \frac{\frac{v_y v_y r_y}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}, \frac{\frac{v_x v_x r_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + \frac{\frac{v_y v_y r_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + v_z + \frac{\frac{v_z v_z r_z}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} \right) = \bar{r}\frac{\frac{\bar{v}\cdot\bar{v}}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + \bar{v} = \nabla(\bar{v}\cdot\bar{r})$$

It would probably have been easier to take the dot product of $(\bar{v}\cdot\bar{r})$ and then take the gradient of that.

We then have:

$$\nabla\phi(P,t) = \nabla\frac{q}{4\pi\epsilon_o\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]_{ret}} = \frac{q}{4\pi\epsilon_o}\nabla\frac{1}{\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]_{ret}} = \frac{q}{4\pi\epsilon_o}\left[-\frac{1}{\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]^2_{ret}}\left[\nabla r - \frac{1}{c}\nabla(\bar{v}\cdot\bar{r})\right]\right]$$

$$\nabla\phi = -\frac{q}{4\pi\epsilon_o}\frac{1}{\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]^2_{ret}}\left[\frac{\bar{r}}{r} + \frac{\bar{r}}{r}\frac{\frac{\bar{v}\cdot\bar{r}}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} - \frac{1}{c}\left(\bar{r}\frac{\frac{\bar{v}\cdot\bar{v}}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + \bar{v}\right)\right]$$

$$\nabla\phi = -\frac{q}{4\pi\epsilon_o}\frac{1}{\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]^2_{ret}}\left[\frac{\bar{r}}{r} + \frac{\frac{\bar{r}}{r}\frac{(\bar{v}\cdot\bar{r})}{cr} - \frac{\bar{r}}{c}\frac{(\bar{v}\cdot\bar{v})}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} - \frac{\bar{v}}{c}\right]$$

4.2 Vector potential

Next, we look at $\frac{\partial\bar{A}}{\partial t}$:

$$\frac{\partial\bar{A}}{\partial t}(P,t) = \frac{\partial}{\partial t}\frac{q\bar{v}}{4\pi\epsilon_o c^2\left[r-\frac{(\bar{v}\cdot\bar{r})}{c}\right]_{ret}} = \frac{q\bar{v}}{4\pi\epsilon_o c^2}\frac{\partial}{\partial t}\frac{1}{\left[r-\frac{(\bar{v}\cdot\bar{r})}{c}\right]_{ret}}$$

$$= -\frac{q\bar{v}}{4\pi\epsilon_o c^2}\frac{1}{\left[r-\frac{(\bar{v}\cdot\bar{r})}{c}\right]^2_{ret}}\left[\frac{\partial r}{\partial t} - \frac{\partial}{\partial t}\frac{(\bar{v}\cdot\bar{r})}{c}\right]$$

We have $r = \sqrt{r_x^2 + r_y^2 + r_z^2}$ and then:

$$\frac{\partial r}{\partial t} = \frac{1}{r}\left(-\frac{r_x v_x}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} - \frac{r_y v_y}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} - \frac{r_z v_z}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right) = -\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

For $\frac{\partial}{\partial t}\frac{(\bar{v}\cdot\bar{r})}{c}$, we have:

$$\frac{\partial}{\partial t}\frac{(\bar{v}\cdot\bar{r})}{c} = \frac{\bar{v}}{c}\cdot\frac{\partial\bar{r}}{\partial t} = \frac{\bar{v}}{c}\cdot\left(-\frac{v_x}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}, -\frac{v_y}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}, -\frac{v_z}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)$$

$$= -\frac{1}{c}\left(\frac{v_x v_x}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}, \frac{v_y v_y}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}, \frac{v_z v_z}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right) = -\frac{1}{c}\frac{\bar{v}\cdot\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

We then obtain:

$$\frac{\partial\bar{A}}{\partial t}(P,t) = \frac{\partial}{\partial t}\frac{q\bar{v}}{4\pi\epsilon_o c^2\left[r-\frac{(\bar{v}\cdot\bar{r})}{c}\right]_{ret}} = \frac{q\bar{v}}{4\pi\epsilon_o c^2}\frac{\partial}{\partial t}\frac{1}{\left[r-\frac{(\bar{v}\cdot\bar{r})}{c}\right]_{ret}}$$

$$= -\frac{q\bar{v}}{4\pi\epsilon_o c^2}\frac{1}{\left[r-\frac{(\bar{v}\cdot\bar{r})}{c}\right]^2_{ret}}\left[\frac{\partial r}{\partial t}-\frac{\partial}{\partial t}\frac{(\bar{v}\cdot\bar{r})}{c}\right]$$

$$\frac{\partial \bar{A}}{\partial t} = \frac{q\bar{v}}{4\pi\epsilon_o c^2}\frac{1}{\left[r-\frac{(\bar{v}\cdot\bar{r})}{c}\right]^2_{ret}}\left[\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{1}{c}\frac{\bar{v}\cdot\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right]$$

4.3 Final electric field equation

We obtain:

$$\bar{E} = -\nabla\phi - \frac{\partial \bar{A}}{\partial t}$$

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\frac{1}{\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]^2_{ret}}\left[\frac{\bar{r}}{r}+\frac{\frac{\bar{r}}{r}\frac{(\bar{v}\cdot\bar{r})}{cr}-\frac{\bar{r}}{c}\frac{(\bar{v}\cdot\bar{v})}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\bar{v}}{c}\right]-\frac{q\bar{v}}{4\pi\epsilon_o c^2}\frac{1}{\left[r-\frac{(\bar{v}\cdot\bar{r})}{c}\right]^2_{ret}}\left[\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{1}{c}\frac{\bar{v}\cdot\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right]$$

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\frac{1}{\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]^2_{ret}}\left[\frac{\bar{r}}{r}+\frac{\frac{\bar{r}}{r}\frac{(\bar{v}\cdot\bar{r})}{cr}-\frac{\bar{r}}{c}\frac{(\bar{v}\cdot\bar{v})}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\bar{v}}{c}-\frac{1}{c^2 r}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+\frac{1}{c^3}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right]$$

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\frac{1}{\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]^2_{ret}}\left[\left(\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right)+\frac{(\bar{v}\cdot\bar{r})}{cr}\frac{\left(\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right)}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}-\frac{(\bar{v}\cdot\bar{v})}{c^2}\frac{\left(\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right)}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}\right]$$

We will look at obtaining this same expression from the book equation.

4.4 Book version

We have originally:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{\bar{e_r}}{r^2}+\frac{r}{c}\frac{d}{dt}\left(\frac{\bar{e_r}}{r^2}\right)+\frac{1}{c^2}\frac{d^2}{dt^2}\bar{e_r}\right]$$

So we can take the derivative in the brackets. We can start with $\frac{d}{dt}\left(\frac{\bar{e_r}}{r^2}\right)$

$$\frac{d}{dt}\left(\frac{\bar{e_r}}{r^2}\right) = \frac{d}{dt}\left(\frac{\bar{r}}{r^3}\right) = \frac{1}{r^3}\frac{d}{dt}(\bar{r}) - 3\frac{\bar{r}}{r^4}\frac{d}{dt}(r)$$

We then have:

$$\frac{d}{dt}(\bar{r}) = \left(-\frac{v_x}{1-\frac{\bar{v}\cdot\bar{r}}{cr}},-\frac{v_y}{1-\frac{\bar{v}\cdot\bar{r}}{cr}},-\frac{v_z}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right) = -\frac{\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

$$\frac{d}{dt}(r) = \frac{1}{r}\left(-\frac{v_x r_x}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{v_y r_y}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{v_z r_z}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right) = -\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

Then:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{\bar{e_r}}{r^2}+\frac{r}{c}\frac{d}{dt}\left(\frac{\bar{e_r}}{r^2}\right)+\frac{1}{c^2}\frac{d^2}{dt^2}\bar{e_r}\right]$$

$$\frac{q}{4\pi\epsilon_o}\left[\frac{\bar{e_r}}{r^2}+\frac{r}{c}\left(-\frac{1}{r^3}\frac{\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+3\frac{\bar{r}}{r^5}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)+\frac{1}{c^2}\frac{d^2}{dt^2}\bar{e_r}\right]$$

Next, we look at $\frac{d^2}{dt^2}\bar{e_r}$

$$\frac{d^2}{dt^2}\bar{e_r}=\frac{d^2}{dt^2}\left(\frac{\bar{r}}{r}\right)=\frac{d}{dt}\frac{d}{dt}\left(\frac{\bar{r}}{r}\right)=\frac{d}{dt}\left(-\frac{1}{r}\frac{\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+\frac{\bar{r}}{r^3}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)=\frac{d}{dt}\left[\frac{1}{r}\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\left(-\bar{v}+\frac{\bar{r}}{r^2}(\bar{v}\cdot\bar{r})\right)\right]$$

$$=\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\left(-\bar{v}+\frac{\bar{r}}{r^2}(\bar{v}\cdot\bar{r})\right)\frac{d}{dt}\left(\frac{1}{r}\right)+\frac{1}{r}\left(-\bar{v}+\frac{\bar{r}}{r^2}(\bar{v}\cdot\bar{r})\right)\frac{d}{dt}\left(\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)+\frac{1}{r}\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\frac{d}{dt}\left(\frac{\bar{r}}{r^2}(\bar{v}\cdot\bar{r})\right)$$

We will look at all derivatives individually, so we start with:

$$\frac{d}{dt}\left(\frac{1}{r}\right)=-\frac{1}{r^2}\frac{d}{dt}r=\frac{1}{r^2}\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

$$\frac{d}{dt}\left(\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)=-\frac{1}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}\frac{d}{dt}\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)=\frac{1}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}\frac{d}{dt}\left(\frac{\bar{v}\cdot\bar{r}}{cr}\right)$$

$$\frac{d}{dt}\left(\frac{\bar{v}\cdot\bar{r}}{cr}\right)=\frac{1}{r}\frac{d}{dt}\frac{(\bar{v}\cdot\bar{r})}{c}-\frac{\bar{v}\cdot\bar{r}}{cr^2}\frac{d}{dt}(r)$$

$$=-\frac{1}{cr}\frac{\bar{v}\cdot\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+\frac{\bar{v}\cdot\bar{r}}{cr^2}\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

$$\frac{d}{dt}\left(\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)=\frac{1}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}\left(-\frac{1}{cr}\frac{\bar{v}\cdot\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+\frac{\bar{v}\cdot\bar{r}}{cr^2}\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)$$

$$\frac{d}{dt}\left(\frac{\bar{r}}{r^2}(\bar{v}\cdot\bar{r})\right)=\frac{\bar{v}\cdot\bar{r}}{r^2}\frac{d}{dt}(\bar{r})-2\frac{\bar{r}}{r^3}(\bar{v}\cdot\bar{r})\frac{d}{dt}(r)+\frac{\bar{r}}{r^2}\frac{d}{dt}(\bar{v}\cdot\bar{r})$$

$$=-\frac{\bar{v}\cdot\bar{r}}{r^2}\frac{\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+2\frac{\bar{r}}{r^3}(\bar{v}\cdot\bar{r})\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\bar{r}}{r^2}\frac{(\bar{v}\cdot\bar{v})}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$$

We then have:

$$\frac{d^2}{dt^2}\bar{e_r}=\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\left(-\bar{v}+\frac{\bar{r}}{r^2}(\bar{v}\cdot\bar{r})\right)\frac{d}{dt}\left(\frac{1}{r}\right)+\frac{1}{r}\left(-\bar{v}+\frac{\bar{r}}{r^2}(\bar{v}\cdot\bar{r})\right)\frac{d}{dt}\left(\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)+\frac{1}{r}\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\frac{d}{dt}\left(\frac{\bar{r}}{r^2}(\bar{v}\cdot\bar{r})\right)$$

$$=\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\left(-\bar{v}+\frac{\bar{r}}{r^2}(\bar{v}\cdot\bar{r})\right)\left(\frac{1}{r^2}\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)$$

$$+\frac{1}{r}\left(-\bar{v}+\frac{\bar{r}}{r^2}(\bar{v}\cdot\bar{r})\right)\left[\frac{1}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}\left(-\frac{1}{cr}\frac{\bar{v}\cdot\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+\frac{\bar{v}\cdot\bar{r}}{cr^2}\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)\right]$$

$$+\frac{1}{r}\frac{1}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\left(-\frac{\bar{v}\cdot\bar{r}}{r^2}\frac{\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}+2\frac{\bar{r}}{r^3}(\bar{v}\cdot\bar{r})\frac{1}{r}\frac{\bar{v}\cdot\bar{r}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}-\frac{\bar{r}}{r^2}\frac{(\bar{v}\cdot\bar{v})}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\right)$$

$$= -\frac{1}{r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{cr^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{cr^4}\frac{\bar{v}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}$$
$$+ \frac{1}{cr^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + 2\frac{1}{r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} - \frac{1}{r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}$$

$$= -2\frac{1}{r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + 3\frac{1}{r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{cr^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{cr^4}\frac{\bar{v}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}$$
$$+ \frac{1}{cr^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}$$

We then have:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{\bar{e_r}}{r^2} + \frac{r}{c}\frac{d}{dt}\left(\frac{\bar{e_r}}{r^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\bar{e_r}\right]$$

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{\bar{r}}{r^3} - \frac{1}{cr^2}\frac{\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + 3\frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} - 2\frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + 3\frac{1}{c^2r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right.$$
$$\left. - \frac{1}{c^3r^4}\frac{\bar{v}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^3r^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^3r^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}\right]$$

We can multiply the third, fourth, and last term in brackets on the RHS by $\frac{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}$:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{\bar{r}}{r^3} - \frac{1}{cr^2}\frac{\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + 3\frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} - 3\frac{1}{c^2r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} - 2\frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right.$$
$$+ 2\frac{1}{c^3r^4}\frac{\bar{v}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + 3\frac{1}{c^2r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^3r^4}\frac{\bar{v}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}$$
$$\left. - \frac{1}{c^3r^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^3r^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^3r^4}\frac{\bar{r}(\bar{v}\cdot\bar{v})(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

This simplifies to:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{\bar{r}}{r^3} - \frac{1}{cr^2}\frac{\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}} + 3\frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} - 2\frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^3r^4}\frac{\bar{v}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right.$$
$$\left. + \frac{1}{c^3r^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

We see that we have terms with $\bar{v}, \bar{r}, \bar{v}\cdot\bar{r}, and\ \bar{v}\cdot\bar{v}$. So we can try to group these by multiplying again by $\frac{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}$. We multiply the second term in brackets on the RHS by $\frac{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}$. We obtain:

$$-\frac{1}{cr^2}\frac{\bar{v}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}\frac{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}=\left(-\frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+\frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right)\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)$$
$$=-\frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+2\frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}-\frac{1}{c^3r^4}\frac{\bar{v}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}$$

$$\bar{E}=\frac{q}{4\pi\epsilon_o}\left[\frac{\bar{r}}{r^3}-\frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+2\frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}-\frac{1}{c^3r^4}\frac{\bar{v}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+3\frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}\right.$$
$$-2\frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+\frac{1}{c^3r^4}\frac{\bar{v}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+\frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+\frac{1}{c^3r^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}$$
$$\left.-\frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

This simplifies to:

$$\bar{E}=\frac{q}{4\pi\epsilon_o}\left[\frac{\bar{r}}{r^3}-\frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+3\frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}+\frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+\frac{1}{c^3r^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right.$$
$$\left.-\frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

Multiply the first term in brackets by $\frac{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}$ and we obtain:

$$\frac{\bar{r}}{r^3}\frac{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}=\left(\frac{1}{r^3}\frac{\bar{r}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}-\frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}\right)\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)$$
$$=\frac{1}{r^3}\frac{\bar{r}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}-2\frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}+\frac{1}{c^2r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}$$

$$\bar{E}=\frac{q}{4\pi\epsilon_o}\left[\frac{1}{r^3}\frac{\bar{r}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}-2\frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}+\frac{1}{c^2r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}-\frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+3\frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}\right.$$
$$\left.+\frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+\frac{1}{c^3r^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}-\frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

This simplifies to:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{1}{r^3}\frac{\bar{r}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{c^2r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} - \frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^3r^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

Multiplying the second term in brackets by $\frac{1-\frac{\bar{v}\cdot\bar{r}}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$ gives:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{1}{r^3}\frac{\bar{r}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{c^2r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^3r^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^3r^6}\frac{\bar{r}(\bar{v}\cdot\bar{r})^3}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

Which simplifies to:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{1}{r^3}\frac{\bar{r}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{c^2r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

Multiplying the fourth term by $\frac{1-\frac{\bar{v}\cdot\bar{r}}{cr}}{1-\frac{\bar{v}\cdot\bar{r}}{cr}}$ gives:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{1}{r^3}\frac{\bar{r}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} + \frac{1}{c^2r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^5}\frac{\bar{r}(\bar{v}\cdot\bar{r})^2}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

Which simplifies to:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{1}{r^3}\frac{\bar{r}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} - \frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

In the brackets, we will add and subtract $\frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}$:

$$\bar{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{1}{r^3}\frac{\bar{r}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2} - \frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} + \frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3} - \frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

We can combine the second and third term in brackets on the RHS:

$$-\frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+\frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}=-\frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)=-\frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}$$

We then have:

$$\bar{E}=\frac{q}{4\pi\epsilon_o}\left[\frac{1}{r^3}\frac{\bar{r}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}-\frac{1}{cr^2}\frac{\bar{v}}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}-\frac{1}{c^2r^3}\frac{\bar{v}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+\frac{1}{cr^4}\frac{\bar{r}(\bar{v}\cdot\bar{r})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}+\frac{1}{c^3r^2}\frac{\bar{v}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right.$$
$$\left.-\frac{1}{c^2r^3}\frac{\bar{r}(\bar{v}\cdot\bar{v})}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^3}\right]$$

This simplifies down to:

$$\bar{E}=\frac{q}{4\pi\epsilon_o}\frac{1}{r^2}\frac{1}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)^2}\left[\left(\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right)+\frac{(\bar{v}\cdot\bar{r})}{cr}\frac{\left(\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right)}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}-\frac{(\bar{v}\cdot\bar{v})}{c^2}\frac{\left(\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right)}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}\right]$$

This compares this to my derivation:

$$\bar{E}=\frac{q}{4\pi\epsilon_o}\frac{1}{\left[r-\left(\frac{\bar{v}\cdot\bar{r}}{c}\right)\right]^2_{ret}}\left[\left(\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right)+\frac{(\bar{v}\cdot\bar{r})}{cr}\frac{\left(\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right)}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}-\frac{(\bar{v}\cdot\bar{v})}{c^2}\frac{\left(\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right)}{\left(1-\frac{\bar{v}\cdot\bar{r}}{cr}\right)}\right]$$

We see that we have the same result.